SyDe114 Linear Algebra: Mini-Test 6

DUE: 30 June 2005

V is the 4-dimensional binary vector space, i.e. the vector space consisting of all (a,b,c,d) where a,b,c,d are 0 or 1. The linear transformation $T : V \to V$ is defined by the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

What are the kernel and image of T? Find $[T]_\beta$, the matrix representation of T with respect to the basis $\beta = \{u_1, u_2, u_3, u_4\}$ where

$$u_1 = (1,1,1,1) \qquad u_2 = (1,1,1,0) \qquad u_3 = (1,1,0,0) \qquad u_4 = (1,0,0,0)$$

and demonstrate explicitly that the formula $[T(v)]_\beta = [T]_\beta [v]_\beta$ works for an arbitrary vector in V.

SOLUTION. First reduce[1] A to echelon form:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and conclude that $\text{rank}(A) = \text{rank}(T) = 4$. Therefore the image of T is all of V and $\ker(T) = \{0\}$ (T is non-singular).

To find the matrix representation it's convenient to express a general vector in V as a linear combination of the basis vectors in β. Write $(a,b,c,d) = xu_1 + yu_2 + zu_3 + tu_4$ and solve for the scalars by reducing[2] the augmented matrix to echelon form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 1 & 1 & 1 & 0 & b \\ 1 & 1 & 0 & 0 & c \\ 1 & 0 & 0 & 0 & d \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 1 & 1 & a+d \\ 0 & 0 & 1 & 1 & a+c \\ 0 & 0 & 0 & 1 & a+b \end{array}\right]$$

and using back-substitution[3] to get:

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[1] Arithmetic with binary vectors is incredibly simple, e.g. $(1,0,1,0)+(1,1,1,1) = (1+1, 0+1, 1+1, 0+1) = (0,1,0,1)$. Remember that 1+1=0 in binary.

[2] Echelon form can be arrived at in a single step by re-ordering the last three rows.

[3] Again binary arithmetic makes this very simple: $a + a = 0$ etc. The sum of an even number of the same constant is 0, and an odd number sums to the constant.

$$\begin{aligned} t &= a+b \\ z &= (a+c)+t = (a+c)+(a+b) = b+c \\ y &= (a+d)+z+t = (a+d)+(b+c)+(a+b) = c+d \\ x &= a+y+z+t = a+(c+d)+(b+c)+(a+b) = d \end{aligned}$$

To summarize[4] we have:

$$(a,b,c,d) = du_1 + (c+d)u_2 + (b+c)u_3 + (a+b)u_4$$

Now to get the required matrix representation of T we evaluate:[5]

$$\begin{array}{lclcccccccc} T(u_1) &=& (0,0,0,1) &=& u_1 &+& u_2 & & & & \\ T(u_2) &=& (0,1,0,1) &=& u_1 &+& u_2 &+& u_3 &+& u_4 \\ T(u_3) &=& (1,0,1,0) &=& & & u_2 &+& u_3 &+& u_4 \\ T(u_4) &=& (1,1,0,1) &=& u_1 &+& u_2 &+& u_3 & & \end{array}$$

and arrange the β coordinates for each image vector as the columns of a matrix:

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Finally, the coordinates[6] of $T(v)$ with respect to β are given by:

$$\begin{aligned} T(v) &= (a+c, a+b+c+d, b+c, a+b+c) \\ &= (a+b+c)u_1 + ((b+c)+(a+b+c))u_2 + ((a+b+c+d)+(b+c))u_3 \\ &\quad + ((a+c)+(a+b+c+d))u_4 \\ &= (a+b+c)u_1 + au_2 + (a+d)u_3 + (b+d)u_4 \end{aligned}$$

We can now verify $[T(v)]_\beta = [T]_\beta [v]_\beta$ by writing the matrix equation:

$$\begin{bmatrix} a+b+c \\ a \\ a+d \\ b+d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} d \\ c+d \\ c+b \\ a+b \end{bmatrix}$$

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[4] At this point you should check your answer to confirm there are no blunders.

[5] The first calculation in each line below comes from multiplying A by each (column) vector u_i to evaluate $T(u_i)$. The second calculation in each line comes from expressing the result as a linear combination of the u_i's using your formula found above.

[6] Here you can make use of your convenient general linear combination formula again.